Some notes on Euler Angles.

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1 Removing the rotors from the exponentials.

In [Doran and Lasenby(2003)] section 2.7.5 the Euler angle formula is developed for $\{z, x', z''\}$ axis rotations by $\{\phi, \theta, \psi\}$ respectively.

Other than a few details the derivation is pretty straightforward. Equation 2.153 would be clearer with a series expansion hint like

$$\begin{aligned}
\exp(R\alpha i R^\dagger) &= \sum_k \frac{1}{k!}(R\alpha i R^\dagger)^k \\
&= \sum_k \frac{1}{k!} R(\alpha i)^k R^\dagger \\
&= R\exp(\alpha i)R^\dagger
\end{aligned}$$

where i is a bivector, and R is a rotor ($RR^\dagger = 1$).

The first rotation is straightforward, by an angle ϕ around the z axis

$$R_\phi(x) = \exp(-Ie_3\phi/2)x\exp(Ie_3\phi/2) = R_\phi x R_\phi^\dagger$$

The next rotation is around the transformed x axis, for which the rotational plane is

$$IR_\phi e_1 R_\phi^\dagger = R_\phi I e_1 R_\phi^\dagger$$

So the rotor for this plane by angle θ is

$$\begin{aligned} R_\theta &= \exp(R_\phi I e_1 R_\phi^\dagger) \\ &= R_\phi \exp(I e_1 \theta/2) R_\phi^\dagger, \end{aligned}$$

resulting in the composite rotor

$$\begin{aligned} R_{\theta\phi} &= R_\theta R_\phi \\ &= R_\phi \exp(I e_1 \theta/2) R_\phi^\dagger R_\phi \\ &= R_\phi \exp(I e_1 \theta/2) \\ &= \exp\left(-I e_3 \phi/2\right) \exp(-I e_1 \theta/2) \end{aligned}$$

The composition of the simple rotation around the e_3 axis, followed by the rotation around the e_1' axis ends up as a product of rotors around the original e_1 and e_3 axis, but curiously enough in inverted order.

2 Expanding the rotor product.

Completing the calculation outlined above follows in the same fashion. The end result is that the composite Euler rotations has the following rotor form

$$\begin{aligned} R(x) &= RxR^\dagger \\ R &= \exp(-e_{12}\phi/2) \exp(-e_{23}\theta/2) \exp(-e_{12}\psi/2) \end{aligned}$$

Then there are notes saying this is easier to visualize and work with than the equivalent matrix formula. Let's see what the equivalent matrix formula is. First calculate the rotor action on $\mathbf{e}_1$

$$
\begin{aligned}
R\mathbf{e}_1R^\dagger &= e^{-e_{12}\phi/2}e^{-e_{23}\theta/2}e^{-e_{12}\psi/2}\mathbf{e}_1 e^{e_{12}\psi/2}e^{e_{23}\theta/2}e^{e_{12}\phi/2} \\
&= e^{-e_{12}\phi/2}e^{-e_{23}\theta/2}\mathbf{e}_1 e^{e_{12}\psi}e^{e_{23}\theta/2}e^{e_{12}\phi/2} \\
&= e^{-e_{12}\phi/2}e^{-e_{23}\theta/2}(\mathbf{e}_1 C_\psi + \mathbf{e}_2 S_\psi)e^{e_{23}\theta/2}e^{e_{12}\phi/2} \\
&= e^{-e_{12}\phi/2}(\mathbf{e}_1 C_\psi + \mathbf{e}_2 S_\psi e^{e_{23}\theta})e^{e_{12}\phi/2} \\
&= e^{-e_{12}\phi/2}(\mathbf{e}_1 C_\psi + \mathbf{e}_2 S_\psi C_\theta + \mathbf{e}_3 S_\psi S_\theta)e^{e_{12}\phi/2} \\
&= (\mathbf{e}_1 C_\psi + \mathbf{e}_2 S_\psi C_\theta)e^{e_{12}\phi} + \mathbf{e}_3 S_\psi S_\theta \\
&= \mathbf{e}_1(C_\psi C_\phi - S_\psi C_\theta S_\phi) + \mathbf{e}_2(C_\psi S_\phi + S_\psi C_\theta C_\phi) + \mathbf{e}_3 S_\psi S_\theta
\end{aligned}
$$

Now $\mathbf{e}_2$:

$$
\begin{aligned}
R\mathbf{e}_2R^\dagger &= e^{-e_{12}\phi/2}e^{-e_{23}\theta/2}e^{-e_{12}\psi/2}\mathbf{e}_2 e^{e_{12}\psi/2}e^{e_{23}\theta/2}e^{e_{12}\phi/2} \\
&= e^{-e_{12}\phi/2}e^{-e_{23}\theta/2}(\mathbf{e}_2 C_\psi - \mathbf{e}_1 S_\psi)e^{e_{23}\theta/2}e^{e_{12}\phi/2} \\
&= (\mathbf{e}_2 C_\psi C_\theta - \mathbf{e}_1 S_\psi)e^{e_{12}\phi} + \mathbf{e}_3 C_\psi S_\theta \\
&= \mathbf{e}_1(-C_\psi C_\theta S_\phi - S_\psi C_\phi) + \mathbf{e}_2(-S_\psi S_\phi + C_\psi C_\theta C_\phi) + \mathbf{e}_3 C_\psi S_\theta
\end{aligned}
$$

And finally $\mathbf{e}_3$

$$
\begin{aligned}
R\mathbf{e}_3R^\dagger &= e^{-e_{12}\phi/2}e^{-e_{23}\theta/2}e^{-e_{12}\psi/2}\mathbf{e}_3 e^{e_{12}\psi/2}e^{e_{23}\theta/2}e^{e_{12}\phi/2} \\
&= e^{-e_{12}\phi/2}\mathbf{e}_3 e^{e_{23}\theta}e^{e_{12}\phi/2} \\
&= e^{-e_{12}\phi/2}(\mathbf{e}_3 C_\theta - \mathbf{e}_2 S_\theta)e^{e_{12}\phi/2} \\
&= -\mathbf{e}_2 S_\theta e^{e_{12}\phi} + \mathbf{e}_3 C_\theta \\
&= \mathbf{e}_1 S_\theta S_\phi - \mathbf{e}_2 S_\theta C_\phi + \mathbf{e}_3 C_\theta
\end{aligned}
$$

This can now be assembled into matrix form

$$
\begin{bmatrix} R(x)\cdot\mathbf{e}_1 \\ R(x)\cdot\mathbf{e}_2 \\ R(x)\cdot\mathbf{e}_3 \end{bmatrix}
=
\begin{bmatrix}
(R\mathbf{e}_1R^\dagger)\cdot\mathbf{e}_1 & (R\mathbf{e}_2R^\dagger)\cdot\mathbf{e}_1 & (R\mathbf{e}_3R^\dagger)\cdot\mathbf{e}_1 \\
(R\mathbf{e}_1R^\dagger)\cdot\mathbf{e}_2 & (R\mathbf{e}_2R^\dagger)\cdot\mathbf{e}_2 & (R\mathbf{e}_3R^\dagger)\cdot\mathbf{e}_2 \\
(R\mathbf{e}_1R^\dagger)\cdot\mathbf{e}_3 & (R\mathbf{e}_2R^\dagger)\cdot\mathbf{e}_3 & (R\mathbf{e}_3R^\dagger)\cdot\mathbf{e}_3
\end{bmatrix}
\begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix}
= \mathbf{R}\mathbf{x}
$$

Therefore we have the composite matrix form for the Euler angle rotations

$$
\mathbf{R} =
\begin{bmatrix}
C_\psi C_\phi - S_\psi C_\theta S_\phi & -S_\psi C_\phi - C_\psi C_\theta S_\phi & S_\theta S_\phi \\
C_\psi S_\phi + S_\psi C_\theta C_\phi & -S_\psi S_\phi + C_\psi C_\theta C_\phi & -S_\theta C_\phi \\
S_\psi S_\theta & C_\psi S_\theta & C_\theta
\end{bmatrix}
$$

Lots of opportunity to make sign errors here. Let's check with matrix multiplication, which should give the same result

3 With composition of rotation matrixes (done wrong, but with discussion and required correction).

$$
\begin{aligned}
R(x) &= \mathbf{R}\mathbf{x} && (1)\\
&= \mathbf{R}_{\psi\mathbf{e}_3}\mathbf{R}_{\theta\mathbf{e}_1}\mathbf{R}_{\phi\mathbf{e}_3}\mathbf{x} && (2)
\end{aligned}
$$

Now, that first rotation is

$$
\begin{aligned}
R_\phi(x) &= e^{-e_{12}\phi/2}(\mathbf{e}_i x^i)e^{e_{12}\phi/2} \\
&= (\mathbf{e}_1 x^1 + \mathbf{e}_2 x^2)e^{e_{12}\phi} + \mathbf{e}_3 x^3 \\
&= x^1(\mathbf{e}_1 \cos\phi + \mathbf{e}_2 \sin\phi) + x^2(\mathbf{e}_2 \cos\phi - \mathbf{e}_1 \sin\phi) + x^3\mathbf{e}_3 \\
&= \mathbf{e}_1(x^1 \cos\phi - x^2 \sin\phi) + \mathbf{e}_2(x^1 \sin\phi + x^2 \cos\phi) + \mathbf{e}_3 x^3
\end{aligned}
$$

Which has the expected matrix form

$$
\mathbf{R}_{\phi\mathbf{e}_3}\mathbf{x} =
\begin{bmatrix}
\cos\phi & -\sin\phi & 0 \\
\sin\phi & \cos\phi & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x^1 \\ x^2 \\ x^3
\end{bmatrix}
$$

Using $C_x = \cos(x)$, and $S_x = \sin(x)$ for brievity, the composite rotation is

$$
\begin{aligned}
\mathbf{R}_{\psi\mathbf{e}_3}\mathbf{R}_{\theta\mathbf{e}_1}\mathbf{R}_{\phi\mathbf{e}_3} &=
\begin{bmatrix}
C_\psi & -S_\psi & 0 \\
S_\psi & C_\psi & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & C_\theta & -S_\theta \\
0 & S_\theta & C_\theta
\end{bmatrix}
\begin{bmatrix}
C_\phi & -S_\phi & 0 \\
S_\phi & C_\phi & 0 \\
0 & 0 & 1
\end{bmatrix} \\
&=
\begin{bmatrix}
C_\psi & -S_\psi C_\theta & S_\psi S_\theta \\
S_\psi & C_\psi C_\theta & -C_\psi S_\theta \\
0 & S_\theta & C_\theta
\end{bmatrix}
\begin{bmatrix}
C_\phi & -S_\phi & 0 \\
S_\phi & C_\phi & 0 \\
0 & 0 & 1
\end{bmatrix} \\
&=
\begin{bmatrix}
C_\psi C_\phi - S_\psi C_\theta S_\phi & -C_\psi S_\phi - S_\psi C_\theta C_\phi & S_\psi S_\theta \\
S_\psi C_\phi + C_\psi C_\theta S_\phi & -S_\psi S_\phi + C_\psi C_\theta C_\phi & -C_\psi S_\theta \\
S_\theta S_\phi & S_\theta C_\phi & C_\theta
\end{bmatrix}
\end{aligned}
$$

This is different from the rotor generated result above, although with a ϕ, and ψ interchange things appear to match?

$$
\begin{bmatrix}
C_\phi C_\psi - S_\phi C_\theta S_\psi & -S_\phi C_\psi - C_\phi C_\theta S_\psi & S_\theta S_\psi \\
C_\phi S_\psi + S_\phi C_\theta C_\psi & -S_\phi S_\psi + C_\phi C_\theta C_\psi & -S_\theta C_\psi \\
S_\phi S_\theta & C_\phi S_\theta & C_\theta
\end{bmatrix}
$$

Where is the mistake? I suspect it is in the matrix formulation, where the plain old rotations for the axis were multiplied. Because the rotations need to be along the transformed axis I bet there is a reversion of matrix products as there was an reversion of rotors? How would one show if this is the case?

What's needed is more careful treatment of the rotation about the transformed axis. Considering the first, for a rotation around the $\mathbf{e}_1' = (C_\phi, S_\phi, 0)$ axis. From [Joot()] we have the rotation matrix for a θ rotation about an arbitrary normal $\mathbf{n} = (n_1, n_2, n_3)$

$$
\begin{aligned}
R_\theta &= \begin{bmatrix}
C_\theta(1-n_1^2)+n_1^2 & n_1 n_2(1-C_\theta)-n_3 S_\theta & n_1 n_3(1-C_\theta)+n_2 S_\theta \\
n_1 n_2(1-C_\theta)+n_3 S_\theta & C_\theta(1-n_2^2)+n_2^2 & n_2 n_3(1-C_\theta)-n_1 S_\theta \\
n_1 n_3(1-C_\theta)-n_2 S_\theta & n_2 n_3(1-C_\theta)+n_1 S_\theta & C_\theta(1-n_3^2)+n_3^2
\end{bmatrix} \\
&= \begin{bmatrix}
C_\theta(1-{C_\phi}^2)+{C_\phi}^2 & C_\phi S_\phi(1-C_\theta) & S_\phi S_\theta \\
C_\phi S_\phi(1-C_\theta) & C_\theta(1-{S_\phi}^2)+{S_\phi}^2 & -C_\phi S_\theta \\
-S_\phi S_\theta & C_\phi S_\theta & C_\theta
\end{bmatrix} \\
&= \begin{bmatrix}
C_\theta({S_\phi}^2)+{C_\phi}^2 & C_\phi S_\phi(1-C_\theta) & S_\phi S_\theta \\
C_\phi S_\phi(1-C_\theta) & C_\theta({C_\phi}^2)+{S_\phi}^2 & -C_\phi S_\theta \\
-S_\phi S_\theta & C_\phi S_\theta & C_\theta
\end{bmatrix}
\end{aligned}
$$

Now the composite rotation for the sequence of ϕ, and θ rotations about the z, and x' axis is

$$
\begin{aligned}
\mathbf{R}_{\phi\mathbf{e}_3,\theta\mathbf{e}_1'} &= \begin{bmatrix}
C_\theta({S_\phi}^2)+{C_\phi}^2 & C_\phi S_\phi(1-C_\theta) & S_\phi S_\theta \\
C_\phi S_\phi(1-C_\theta) & C_\theta({C_\phi}^2)+{S_\phi}^2 & -C_\phi S_\theta \\
-S_\phi S_\theta & C_\phi S_\theta & C_\theta
\end{bmatrix}
\begin{bmatrix}
C_\phi & -S_\phi & 0 \\
S_\phi & C_\phi & 0 \\
0 & 0 & 1
\end{bmatrix} \\
&= \begin{bmatrix}
C_\phi & -C_\theta S_\phi & S_\phi S_\theta \\
S_\phi & C_\theta C_\phi & -C_\phi S_\theta \\
0 & S_\theta & C_\theta
\end{bmatrix} \\
&= \begin{bmatrix}
C_\phi & -S_\phi & 0 \\
S_\phi & C_\phi & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & C_\theta & -S_\theta \\
0 & S_\theta & C_\theta
\end{bmatrix} \\
&= \mathbf{R}_{\phi\mathbf{e}_3}\mathbf{R}_{\theta\mathbf{e}_1}
\end{aligned}
$$

Wow, sure enough the composite rotation matrix is the result of the inverted order product of the two elementary rotation matrixes. The algebra here is fairly messy, so it would not be terribly fun to go one step further using just matrixes that the final triple rotation is not the product of equation 1, but instead requires the matrix product

$$
\begin{aligned}
R(x) &= \mathbf{R}\mathbf{x} && (3) \\
&= \mathbf{R}_{\phi\mathbf{e}_3}\mathbf{R}_{\theta\mathbf{e}_1}\mathbf{R}_{\psi\mathbf{e}_3}\mathbf{x} && (4)
\end{aligned}
$$

Assuming that this is true, the swapping of angles to match the rotor expression is fully accounted for, and it is understood how to correctly do the same calculation in matrix form.

4 Relation to Cayley-Klein parameters.

Exercise 2.9 from [Doran and Lasenby(2003)] is to relate the rotation matrix expressed in terms of Cayley-Klein parameters back to the rotor formulation. That matrix has the look of something that involves the half angles. Use of software to expressing the rotation in terms of the half angle signs and cosines, plus some manually factoring (which could be caried further), produces the following mess

$$
\begin{aligned}
R_{11} &= S_\phi^2 S_\psi^2 - C_\psi^2 S_\phi^2 - C_\phi^2 S_\psi^2 + C_\phi^2 C_\psi^2 - 4C_\phi S_\phi C_\psi S_\psi (C_\theta^2 - S_\theta^2) \\
R_{21} &= 2C_\psi S_\psi (C_\phi^2 - S_\phi^2)(C_\theta^2 - S_\theta^2) + 2C_\phi S_\phi (C_\psi^2 - S_\psi^2) \\
R_{31} &= 4C_\psi C_\theta S_\psi S_\theta \\
R_{12} &= -2C_\phi S_\phi (C_\psi^2 - S_\psi^2)(C_\theta^2 - S_\theta^2) - 2C_\psi S_\psi (C_\phi^2 - S_\phi^2) \\
R_{22} &= (C_\theta^2 - S_\theta^2)(-S_\phi^2 + C_\phi^2)(C_\psi^2 - S_\psi^2) - 4C_\phi S_\phi C_\psi S_\psi \\
R_{32} &= 2C_\theta S_\theta (C_\psi^2 - S_\psi^2) \\
R_{13} &= 4C_\phi C_\theta S_\phi S_\theta \\
R_{23} &= -2C_\theta S_\theta (C_\phi^2 - S_\phi^2) \\
R_{33} &= +C_\theta^2 S_\phi^2 - S_\psi^2 S_\theta^2 - C_\psi^2 S_\theta^2 + C_\phi^2 C_\theta^2
\end{aligned}
$$

It kind of looks like the terms $C_\theta S_\phi$ may be related to these parameters. error prone. A google search for Cayley Klein also verifies that those parameters are expressed in terms of half angle relations, but even with the hint, I wasn't successful getting something tidy out of all this.

An alternate approach is to just expand the rotor, so terms may be grouped before that rotor and its reverse is applied to the object to be rotated. Again in terms of half angle signs and cosines this is

$$
\begin{aligned}
R &= \exp(-e_{12}\phi/2)\exp(-e_{23}\theta/2)\exp(-e_{12}\psi/2) \\
&= (C_\phi C_\psi - S_\phi S_\psi)C_\theta - (C_\psi S_\phi + C_\phi S_\psi)C_\theta \mathbf{e}_{12} \\
&\quad + (C_\phi S_\psi - C_\psi S_\phi)S_\theta \mathbf{e}_{31} - (S_\phi S_\psi + C_\phi C_\psi)S_\theta \mathbf{e}_{23}
\end{aligned}
$$

Grouping terms produces

$$
R = \cos\left(\frac{\theta}{2}\right)\exp\left(\frac{-\mathbf{e}_{12}}{2}(\psi+\phi)\right) + \sin\left(\frac{\theta}{2}\right)\exp\left(\frac{\mathbf{e}_{12}}{2}(\psi-\phi)\right)\mathbf{e}_{32} \tag{5}
$$

Okay... now I see how you naturally get four parameters out of this. Also see why it was hard to get there from the fully expanded rotation product ... it would first be required to group all the ϕ and ψ terms just right in terms of sums and differences.

With

$$
\begin{aligned}
R &= \alpha + \delta\mathbf{e}_{12} + \beta\mathbf{e}_{23} + \gamma\mathbf{e}_{31} \\
\alpha &= \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{1}{2}\left(\psi+\phi\right)\right) \\
\delta &= -\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{1}{2}\left(\psi+\phi\right)\right) \\
\beta &= -\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{1}{2}\left(\psi-\phi\right)\right) \\
\gamma &= \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{1}{2}\left(\psi-\phi\right)\right)
\end{aligned}
$$

By inspection, these have the required property $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$, and multiplying out the rotors yields the rotation matrix

$$
\mathbf{U} = \begin{bmatrix}
-\gamma^2 + \beta^2 - \delta^2 + \alpha^2 & +2\beta\gamma + 2\alpha\delta & +2\delta\beta - 2\alpha\gamma \\
+2\beta\gamma - 2\alpha\delta & +\gamma^2 - \beta^2 - \delta^2 + \alpha^2 & +2\delta\gamma + 2\alpha\beta \\
+2\delta\beta + 2\alpha\gamma & +2\delta\gamma - 2\alpha\beta & -\gamma^2 - \beta^2 + \delta^2 + \alpha^2
\end{bmatrix}
$$

Now that particular choice of sign and permutation of the α, β, γ, and δ isn't at all obvious, and is also arbitrary. Of the 120 different sign and permutation variations that can be tried, this one results in the particular desired matrix **U** from the problem. The process of performing the multiplications was well suited to a symbolic GA calculator and one was written with this and other problems in mind.

[Goldstein(1951)] also treats these Cayley-Klein parameterizations, but does so considerably differently. He has complex parameterizations and quaternion matrix representations, and it will probably be worthwhile to reconsile all of these.

5 Omitted details.

5.1 Cayley Klein details.

Equation 5 was obtained with the following manipulations

$$
\begin{aligned}
R &= (C_\phi C_\psi - S_\phi S_\psi) C_\theta - (C_\psi S_\phi + C_\phi S_\psi) C_\theta \mathbf{e}_{12} \\
&+ (C_\phi S_\psi - C_\psi S_\phi) S_\theta \mathbf{e}_{31} - (S_\phi S_\psi + C_\phi C_\psi) S_\theta \mathbf{e}_{23} \\
&= \cos\left(\frac{1}{2}(\phi+\psi)\right) C_\theta - \sin\left(\frac{1}{2}(\phi+\psi)\right) C_\theta \mathbf{e}_{12} \\
&- \sin\left(\frac{1}{2}(\phi-\psi)\right) S_\theta \mathbf{e}_{31} - \cos\left(\frac{1}{2}(\phi-\psi)\right) S_\theta \mathbf{e}_{23} \\
&= \cos\left(\frac{1}{2}(\psi+\phi)\right) C_\theta - \sin\left(\frac{1}{2}(\psi+\phi)\right) C_\theta \mathbf{e}_{12} \\
&+ \sin\left(\frac{1}{2}(\psi-\phi)\right) S_\theta \mathbf{e}_{31} - \cos\left(\frac{1}{2}(\psi-\phi)\right) S_\theta \mathbf{e}_{23} \\
&= C_\theta \left(\cos\left(\frac{1}{2}(\psi+\phi)\right) - \sin\left(\frac{1}{2}(\psi+\phi)\right) \mathbf{e}_{12}\right) \\
&+ S_\theta \left(\sin\left(\frac{1}{2}(\psi-\phi)\right) \mathbf{e}_{31} - \cos\left(\frac{1}{2}(\psi-\phi)\right) \mathbf{e}_{23}\right) \\
&= C_\theta \exp\left(\frac{-\mathbf{e}_{12}}{2}(\psi+\phi)\right) \\
&+ S_\theta \mathbf{e}_3 \left(\sin\left(\frac{1}{2}(\psi-\phi)\right) \mathbf{e}_1 + \cos\left(\frac{1}{2}(\psi-\phi)\right) \mathbf{e}_2\right) \\
&= C_\theta \exp\left(\frac{-\mathbf{e}_{12}}{2}(\psi+\phi)\right) + S_\theta \exp\left(\frac{\mathbf{e}_{12}}{2}(\psi-\phi)\right) \mathbf{e}_{32}
\end{aligned}
$$

In that last step an arbitrary but convienent decision to write the complex number i as $\mathbf{e}_{12}$ was employed.

References

[Doran and Lasenby(2003)] C. Doran and A.N. Lasenby. *Geometric algebra for physicists*. Cambridge University Press New York, 2003.

[Goldstein(1951)] H. Goldstein. Classical mechanics. 1951.

[Joot()] Peeter Joot. Rotor notes. ”`http://sites.google.com/site/peeterjoot/geometric-algebra/rotor.pdf`”.